

**TUTORIAL-4: B.Tech. Sem II, 24/03/2025 Mathematics-II (Fourier Series)****Prof. Raj Kumar**

**Preamble: (i).**  $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$

**(ii)**  $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$

**(iii)**  $\int u v \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \dots$ ,  $u, v$  are functions of  $x$  and dashes denote differentiation and suffixes denote integration with respect to  $x$

**(iv)** Fourier series for  $f(x)$  in  $c \leq x \leq c + 2\pi$  is  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ,

where  $a_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \, dx$ ,  $a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx \, dx$ ,  $b_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \sin nx \, dx$

**(v)** Fourier series for  $f(x)$  in  $c \leq x \leq c + 2l$  is  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ,

where  $a_0 = \frac{1}{l} \int_c^{c+2l} f(x) \, dx$ ,  $a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \frac{n\pi x}{l} \, dx$ ,  $b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \frac{n\pi x}{l} \, dx$

**(vi)** The half range cosine series in  $0 \leq x \leq l$  is  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

where  $a_0 = \frac{2}{l} \int_0^l f(x) \, dx$ ,  $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} \, dx$ , and half range sine series in  $0 \leq x \leq l$  is

$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$  where  $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$

Q.1 Express  $f(x) = \frac{1}{2}(\pi - x)$  in a Fourier series in the interval  $0 < x < 2\pi$ . Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

**Ans:**  $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$

Q.2 Find the Fourier series to represent the function  $f(x) = |\sin x|$ ,  $-\pi < x < \pi$ .

$$\text{Ans: } |\sin x| = \frac{2}{\pi} - \frac{4}{\pi} \left( \frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \dots, \frac{\cos 2nx}{4n^2 - 1} + \dots \right)$$

Q.3 Express  $f(x) = |x|$ ,  $-\pi < x < \pi$ , as a Fourier series. Hence show that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}. \quad \text{Ans: } |x| = \frac{\pi}{2} - \frac{4}{\pi} \left( \cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right)$$

Q.4 Find the Fourier series for function  $f(x) = x + x^2$ ;  $-\pi < x < \pi$ , Hence show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots \quad \text{Ans: } f(x) = \frac{\pi^2}{3} + 4 \left( \frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots \right) - 2 \left( \frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + \dots \right)$$

Q.5 Expand the function  $f(x) = x \sin x$  as a Fourier series in the interval  $-\pi \leq x \leq \pi$ .

$$\text{Ans: } x \sin x = 1 - \frac{1}{2} \cos x - 2 \left( \frac{\cos 2x}{2^2 - 1} - \frac{\cos 3x}{3^2 - 1} + \frac{\cos 4x}{4^2 - 1} - \dots \right)$$

Q.6 Find the Fourier series to represent the function  $f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 < x < \pi \end{cases}$ . Also deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad \text{Ans: } f(x) = \frac{4k}{\pi} \left( \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)$$

Q.7 Find the Fourier series of  $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ -\pi - x, & 0 < x < \pi \end{cases}$  which is assumed to be periodic with

$$\text{period } 2\pi. \quad \text{Ans: } f(x) = \frac{-\pi}{2} + \frac{4}{\pi} \left( \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \dots \right) + 4 \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \dots \right)$$

Q.8 Obtain Fourier series for function  $f(x) = \begin{cases} \pi x & 0 \leq x \leq 1 \\ \pi(2-x) & 1 \leq x \leq 2 \end{cases}$

$$\text{Ans: } \frac{\pi}{2} - \frac{4}{\pi} \leq \left( \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right)$$

Q.9 Find the Fourier series to represent  $f(x) = x^2 - 2$ ,  $-2 \leq x \leq 2$ .

$$\text{Ans: } x^2 - 2 = -\frac{2}{3} - \frac{16}{\pi^2} \left( \cos \frac{\pi x}{2} - \frac{1}{4} \cos \pi x + \frac{1}{9} \cos \frac{3\pi x}{2} - \dots \right)$$

Q.10 Obtain the half range sine series for  $f(x) = e^x$ ,  $0 < x < 1$ . **Ans:**

$$e^x = 2\pi \sum_{n=1}^{\infty} \frac{n[1 - e(-1)^n]}{1 + n^2\pi^2} \sin n\pi x$$

Q.11 Express  $\sin x$  as a cosine series in  $0 < x < \pi$ . **Ans:**  $\sin x = \frac{2}{\pi} - \frac{4}{\pi} \left( \frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \dots \right)$

Q.12 Obtain the half-range sine series for the function  $f(x) = x^2$  in the interval  $(0, 3)$ .

$$\text{Ans: } f(x) = \sum_{n=1}^{\infty} b_n \frac{\sin n\pi x}{3}, \text{ where } b_n = \frac{18}{n\pi}(-1)^{n+1} + \frac{36}{n^3\pi^3} [(-1)^n - 1]$$